Florian Wickelmaier
Forschungsmethoden und Mathematische Psychologie
Fachbereich Psychologie
Universität Tübingen

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Exercises

Exercise 1. The following table lists the standardized values that three persons P_1, P_2 and P_3 from a sample provide in tests X_1 and X_2:

	X_1	X_2
P_1	2	1
P_2	1	-2
P_3	-2	4

a) Enter the person-specific values as column vectors $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$ and draw them as vectors in the variable space spanned by X_1 and X_2.

b) Is the system of vectors $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$ linearly independent? Justify your decision both graphically and geometrically, as well as formally via the definition of linear independence.

Exercise 2. Given matrices

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 2 \\ 2 & 2 & 3 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 & 1 \\ 2 & -1 \\ 3 & 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix} \text{ and } \mathbf{D} = \begin{pmatrix} 4 & 1 & 0 \\ 0 & 5 & 0 \end{pmatrix}.$$

Identify the sums $\mathbf{A} + \mathbf{B}$, $\mathbf{A} + \mathbf{C}$, $\mathbf{A} + \mathbf{D}$, $\mathbf{B} + \mathbf{D}$ and products $\mathbf{AB}$, $\mathbf{AC}$, $\mathbf{AD}$, $\mathbf{BD}$ that are properly defined. Specify the number of rows and columns of the resulting matrices and calculate them by hand and using R.

Exercise 3. Let $\mathbf{x}$ be an n-vector and $\mathbf{1}$ be the n-vector consisting of ones exclusively.

a) Compute the products $\mathbf{1}'\mathbf{1}$ and $\mathbf{1}\mathbf{1}'$.

b) What do we obtain by computing $\frac{1}{n}\mathbf{1}'\mathbf{x}$?

Exercise 4. Let the vectors $\mathbf{x}, \mathbf{y}$ and the matrix $\mathbf{A}$ be given by

$$\mathbf{x} = \begin{pmatrix} 7 \\ 4 \\ 1 \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} 2 & 7 & 1 \\ 3 & 0 & 4 \end{pmatrix}.$$

Solve the following problems by hand and using R.

a) Calculate (whenever defined) $\mathbf{x}'\mathbf{x}$, $\mathbf{x}\mathbf{x}'$, $\mathbf{x}\mathbf{y}$, $\mathbf{x}'\mathbf{y}$ and $\mathbf{x}\mathbf{y}'$.

b) Specify $\mathbf{A}\mathbf{x}$ and $\mathbf{x}'\mathbf{A}'$.

c) Calculate trace and determinant for $\mathbf{x}'\mathbf{x}$ and $\mathbf{x}\mathbf{x}'$.

d) Are row or column vectors, respectively, in $\mathbf{x}\mathbf{x}'$ linearly independent?

Exercise 5.* The set of all real (2×2) matrices forms a vector space with respect to matrix addition and scalar multiplication. Specify a base for this vector space by showing

a) that every arbitrary matrix

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

with $a_{11}, a_{12}, a_{21}, a_{22} \in \mathbb{R}$ can be expressed as linear combination of the elements of the basis;

b) that the elements of the basis are linearly independent.

Specify the dimension of the $\mathbb{R}$ vector space of the (2×2) matrices.

Exercise 6. Given the matrix

$$\mathbf{A} = \begin{pmatrix} 4 & 5 \\ -2 & 1 \end{pmatrix},$$

solve the following problems by hand and using R.

a) Is $\mathbf{A}$ invertible? Justify why or why not?

b) If it exists, calculate the inverse $\mathbf{A}^{-1}$.

c) Check if $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$

d) Which vectors $\mathbf{x}$ satisfy the equation

$$\mathbf{A}\mathbf{x} = \begin{pmatrix} 2 \\ 6 \end{pmatrix}?$$

Exercise 7. Determine the number a such that the two vectors

$$\mathbf{x} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} -2 \\ a \\ 5 \end{pmatrix}$$

are orthogonal.

Exercise 8. The points in $\mathbb{R}^2$ given by the vectors

$$\mathbf{x} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}, \quad \mathbf{z} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

define a triangle in the coordinate plane. Calculate the result of multiplying these vectors with the matrices

$$\mathbf{A} = \begin{pmatrix} -0.707 & -0.707 \\ 0.707 & -0.707 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix}$$

(multiply in the order matrix-vector).

Illustrate the respective transformations of the triangle graphically, and provide geometric interpretations of the associated linear mappings.