Statistik III

Advanced Statistics

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Rasch Model

Notation

- Let $A = \{a_1, \ldots, a_i, \ldots, a_m\}$ denote the set of persons and $X = \{x_1, \ldots, x_j, \ldots, x_n\}$ the set of (dichotomous) items presented to the persons, which can be answered either correctly or incorrectly
- On an appropriately chosen probability space $\langle \Omega, \mathcal{A}, P \rangle$, for all $i = 1, \ldots, m$ and $j = 1, \ldots, n$ binary random variables are introduced by

$$U_{ij}(\omega) = \begin{cases} 1 & \text{if person } a_i \text{ solves the item } x_j \\ 0 & \text{if person } a_i \text{ does not solve the item } x_j \end{cases}$$

- $\mathbf{U}_i$ and $\mathbf{U}$ denote the random vector assigned to the person a_i and the (total) matrix of random variables, respectively

Rasch Model

Latent trait

- We assume a latent trait that underlies the manifest solution behavior
- The psychometric models built upon this assumption are called latent trait models, and form the class of item response theories
- The latent trait is conceived as person ability and often viewed as a one-dimensional continuum
- In this case, the mapping

$$\theta \colon A \to \mathbb{R}$$

assigns a latent ability value to each of the individuals

Rasch Model

1-parametric logistic model

- Characterization of the items

Mapping	Interpretation
$\varepsilon\colon X \to I\!R$	item difficulty

- Likelihood of the answer of person a_i at item x_j

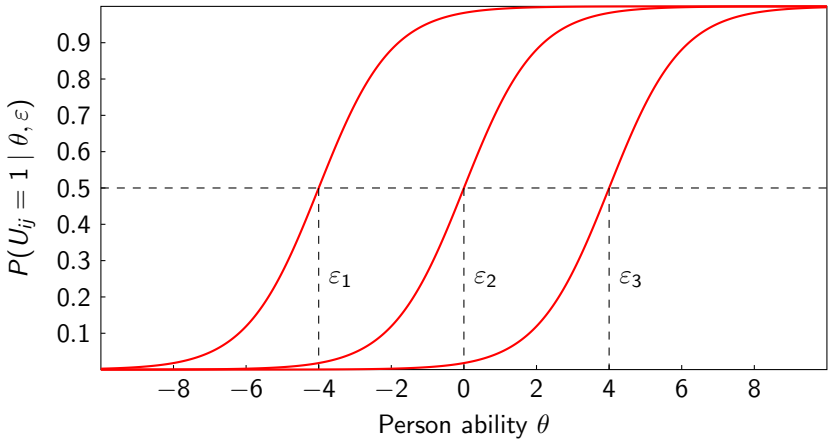
$$P(U_{ij} = 1 \mid \theta_i, \varepsilon_j) = \frac{1}{1 + \exp[-(\theta_i - \varepsilon_j)]}$$

- Alternative formulation of the model equation with item easiness $\beta_j = -\varepsilon_j$ replacing item difficulty

$$P(U_{ij} = 1 \mid \theta_i, \beta_j) = \frac{1}{1 + \exp[-(\theta_i + \beta_j)]}$$

Rasch Model

1-parametric logistic model: Item characteristic curve (ICC)



Stochastic Independence

Assumptions

- Let $\Theta = (\theta_1, \ldots, \theta_m)$ denote the vector of person parameters and $\Xi = (\xi_1, \ldots, \xi_n)$ a general vector of item parameters with $\xi_j = (\xi_{j1}, \ldots, \xi_{jl})$ for all $j = 1, \ldots, n$
- The responses of different persons are stochastically independent

$$P(\mathbf{U} = \mathbf{u} \mid \Theta, \Xi) = \prod_{i=1}^{m} P(\mathbf{U}_i = \mathbf{u}_i \mid \theta_i, \Xi)$$

- For a fixed person the responses are stochastically independent across items

$$P(\mathbf{U}_i = \mathbf{u}_i \mid \theta_i, \Xi) = \prod_{j=1}^{n} P(U_{ij} = u_{ij} \mid \theta_i, \xi_j)$$

Stochastic Independence

Conclusions

- The independence assumptions imply the following decomposition of the probability of the data matrix $\mathbf{u}$

$$P(\mathbf{U} = \mathbf{u} \mid \Theta, \Xi) = \prod_{i=1}^{m} P(\mathbf{U}_i = \mathbf{u}_i \mid \theta_i, \Xi)$$
$$= \prod_{i=1}^{m} \prod_{j=1}^{n} P(U_{ij} = u_{ij} \mid \theta_i, \xi_j)$$

- Under the above assumptions, the probability of the entire data set is the product of the solution probabilities specified by a particular model across all individuals and all items

Stochastic Independence

Problems

- Both the assumption of independence over persons and the local stochastic independence of items given a fixed person may be systematically violated for hierarchically structured data
- When applying a latent trait model, any dependencies that may exist between persons and/or items are ignored
- As a consequence, the parameter estimates may be inconsistent or the estimated error variances may deviate from the "true variances"
- This may also compromise statistical decisions

Generalized Linear Mixed Models

The generalized linear mixed model (GLMM) states that

$$g(E(\mathbf{y}_i \mid \boldsymbol{v}_i)) = \mathbf{X}_i \boldsymbol{\beta} + \mathbf{Z}_i \boldsymbol{v}_i$$

with

- fixed effects $\boldsymbol{\beta}$
- random effects $\boldsymbol{v}_i$
- design matrices $\mathbf{X}_i$ and $\mathbf{Z}_i$
- distributional assumptions
$$\boldsymbol{v}_i \sim N(\mathbf{0}, \boldsymbol{\Sigma}_v) \text{ i.i.d.}$$
- the link function g relating the expected value of the dependent variable (conditional on the random effects) with the linear combination of the predictors

The conditional expected value is the mean of an exponential family distribution, such as of the binomial distribution with $Y_{ij} \sim B(n = 1, \pi_{ij} = E(y_{ij} \mid \boldsymbol{v}_i))$

Rasch Model as GLMM

Specifications

- The link function g is the logit function
- $\mathbf{y}_i$ is the (random) vector of responses of person i for all items (n vector of indicator variables)
- The design matrix $\mathbf{X}_i$ is identical for all individuals i and is given by the identity matrix

$$\mathbf{X}_i = \mathbf{I}_n = \begin{pmatrix} 1 & 0 & \dots & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots & & \vdots \\ 0 & 0 & \dots & 1 & \dots & 0 \\ \vdots & \vdots & & \vdots & & \vdots \\ 0 & 0 & \dots & 0 & \dots & 1 \end{pmatrix}$$

Rasch Model as GLMM

Specifications

- Item easiness parameters are represented as fixed effects $\boldsymbol{\beta}$
- The design matrix $\mathbf{Z}_i$ is identical for all individuals i and is given by the n vector of ones

$$\mathbf{Z}_i = \mathbf{1} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

- The vector $\boldsymbol{v}_i$ of random effects consists of only a single random variable representing the person parameter θ_i (which is a random intercept) and is normally distributed according to $N(0, \sigma_\theta^2)$

Rasch Model as GLMM

- The model equation then reads

$$\begin{pmatrix} \log \frac{\pi_{i1}}{1-\pi_{i1}} \\ \vdots \\ \log \frac{\pi_{ij}}{1-\pi_{ij}} \\ \vdots \\ \log \frac{\pi_{in}}{1-\pi_{in}} \end{pmatrix} = \begin{pmatrix} 1 & \dots & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & & \vdots \\ 0 & \dots & 1 & \dots & 0 \\ \vdots & & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & \dots & 1 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_j \\ \vdots \\ \beta_n \end{pmatrix} + \begin{pmatrix} 1 \\ \vdots \\ 1 \\ \vdots \\ 1 \end{pmatrix} \theta_i$$

with

$$\pi_{ij} = P(y_{ij} = 1 \mid \boldsymbol{\beta}, \theta_i)$$

- For item x_j this provides the basic equation of the Rasch model

$$\log \frac{\pi_{ij}}{1-\pi_{ij}} = \beta_j + \theta_i$$

Example: Artificial Responses

Data (wide format)

```
dat <- read.table("rasch-data.txt")
m <- nrow(dat)
n <- ncol(dat)
names(dat) <- sprintf("I%.2i", 1:10)
```

Reshaping the data into long format

```
dat2 <- reshape(dat, varying = names(dat), v.names = "response",
                idvar = "subj", timevar = "item", times = names(dat),
                direction = "long")
rownames(dat2) <- NULL
```

Example: Artificial Responses

Rasch model with `mirt` package (marginal ML estimation)

$$P(U_{ij} = u_{ij} \mid \theta_i, a_j, b_j) = \frac{1}{1 + \exp[-a_j \cdot (\theta_i - b_j)]}$$

with $a_j = 1$ for all j

```
library(mirt)
m1 <- mirt(dat, model = 1, itemtype = "Rasch")

## Parameters and estimates
pars <- coef(m1, IRTpars = TRUE, as.data.frame = TRUE)
m1.epsi <- pars[paste0(names(dat), ".b"), ]  # item difficulty
m1.theta <- fscores(m1, method = "MAP")[, 1] # ability (empirical Bayes, MAP)
```

Example: Artificial Responses

Rasch model with `lme4` package

```
library(lme4)

## Ability as random intercept
m2 <- glmer(response ~ 0 + item + (1 | subj), dat2, binomial)

m2.epsi  <- -lme4::fixef(m2)      # item difficulty
m2.theta <- ranef(m2)$subj[, 1]   # ability estimates
```

Comparison MML vs. GLMM

```
cbind( mml = m1.epsi - mean(m1.epsi),
       glmm = m2.epsi - mean(m2.epsi))

cbind( mml = m1.theta - mean(m1.epsi),
       glmm = m2.theta - mean(m2.epsi))
```

Example: Artificial Responses

Results (m2)

```
Generalized linear mixed model fit by maximum likelihood
                          (Laplace Approximation) [glmerMod]
 Family: binomial  ( logit )
Formula: response ~ 0 + item + (1 | subj)
   Data: dat.l

     AIC      BIC   logLik deviance df.resid
  2488.1   2549.7  -1233.0   2466.1     1989

Random effects:
 Groups Name        Variance Std.Dev.
 subj   (Intercept) 0.7151   0.8456
Number of obs: 2000, groups:  subj, 200
```

Example: Artificial Responses

Results (`m2`)

```
Fixed effects:
        Estimate Std. Error z value Pr(>|z|)
itemI01  -0.0045     0.1636     0.0    0.978
itemI02   0.5151     0.1671     3.1    0.002 **
itemI03  -0.1674     0.1639    -1.0    0.307
itemI04  -1.5513     0.1961    -7.9    3e-15 ***
itemI05   0.8423     0.1730     4.9    1e-06 ***
itemI06  -0.6208     0.1684    -3.7    2e-04 ***
itemI07   0.9758     0.1763     5.5    3e-08 ***
itemI08  -0.8227     0.1722    -4.8    2e-06 ***
itemI09  -0.3317     0.1649    -2.0    0.044 *
itemI10   1.0032     0.1771     5.7    1e-08 ***
```

Example: Artificial Responses

Comparison MML vs. GLMM (sum-to-zero constraints)

Item parameters

	mml	glmm
itemI01	-0.012	-0.012
itemI02	-0.531	-0.531
itemI03	0.151	0.151
itemI04	1.537	1.535
itemI05	-0.859	-0.859
itemI06	0.604	0.605
itemI07	-0.992	-0.992
itemI08	0.806	0.806
itemI09	0.315	0.316
itemI10	-1.020	-1.019

Person parameters

	mml	glmm
[1,]	0.565	0.555
[2,]	0.274	0.269
[3,]	0.867	0.851
[4,]	-0.888	-0.873
[5,]	0.867	0.851
[6,]	-0.013	-0.013
...		
[196,]	0.274	0.269
[197,]	-0.888	-0.873
[198,]	-1.204	-1.182
[199,]	0.274	0.269
[200,]	-0.299	-0.294

Latent Trait Models as GLMMs

- The flexibility of generalized linear mixed models allows for specifying a variety of models of latent trait theory
- Models the are *not* GLMMs
  - Non-linear models, such as the Birnbaum model (2PL)
  - 3PL model (the latter is also not a pure logit model)
- In a taxonomy of models De Boeck et al. (2011) consider four aspects, referring to the kind of covariates as well as to the kind of effects

Latent Trait Models as GLMMs

Taxonomy

1. Mode of the covariates
   - The covariates can refer to items, to persons, or to pairs of persons and items
2. External versus internal covariates
   - An internal covariate stems from the responses to be modeled, e. g., the number of previous successful responses in a test is an internal (person $\times$ item) covariate for a model that is meant for the item responses
   - An external covariate is external to the item responses to be modeled, e. g., gender is an external (and a person) covariate
3. Fixed versus random effects
   - Fixed effects are unknown constants and do not vary as a function of observational units, whereas random effects do vary across observational units and are drawn from a distribution, which is commonly the normal distribution

Latent Trait Models as GLMMs

Taxonomy

4. Mode of randomness
   - A random effect follows a distribution associated with the population of the observational units one wants to consider
   - An effect can be random across persons, across items, across persons within a person group, across items within an item type, across groups of persons, across groups (types) of items, and even more complicated cases can be constructed
   - It is clear from the nested elements in the list (e. g., persons and person groups) that also multilevel models are a possibility

Example: Verbal Aggression

- Verbal aggression questionnaire administered to first-year psychology students (243 women and 73 men) at the K. U. Leuven (Belgium)
- 24 items developed on the basis of four frustrating situations (De Boeck & Wilson, 2004; data set `VerbAgg` in the `lme4` and the `difR` packages)
- In two situations someone else is to blame (e. g., "A bus fails to stop for me"), and in two situations one is self to blame (e. g., "I am entering a grocery store when it is about to close")
  - Variable `situ` with levels `other`, `self`
- Each of these situations is combined with three behaviors
  - Variable `btype` with levels `curse`, `scold` and `shout`

Example: Verbal Aggression

- These 12 combinations are formulated in two modes (e. g. "I would want to curse" vs. "I would curse")
  - Variable `mode` with levels `want` und `do`
- Response format with three categories "no", "perhaps", "yes" coded dichotomously
  - Variable `r2` with `N` for "no" and `Y` otherwise
- Other covariates
  - `Gender` with `F` for female and `M` for male
  - `Anger` providing a trait score of the person's anger (State-Trait Anger Expression Inventory, STAXI)

Example: Verbal Aggression

Item covariate models

- Rasch model (1PL)

```
m1 <- glmer(r2 ~ 0 + item + (1 | id), VerbAgg, binomial)
```

- Linear logistic test model (LLTM)

```
## Regular LLTM
m2 <- glmer(r2 ~ 0 + btype + mode + situ + (1 | id), VerbAgg, binomial)

## LLTM with homoscedastic error
m3 <- glmer(r2 ~ 0 + btype + mode + situ
                + (1 | id) + (1 | item), VerbAgg, binomial)

## LLTM with heteroscedastic error
m4 <- glmer(r2 ~ 0 + btype + mode + situ
                + (1 | id) + diag(0 + mode | item), VerbAgg, binomial)
```

Example: Verbal Aggression

Person covariate models

- 1PL Models (indicator covariate for persons)

```
## Random persons, fixed items
m1 <- glmer(r2 ~ 0 + item + (1 | id), VerbAgg, binomial)

## Random persons, random items
m5 <- glmer(r2 ~ 1 + (1 | id) + (1 | item), VerbAgg, binomial)

## Fixed persons, fixed items (joint ML)
m6 <- glmer(r2 ~ 0 + id + item + (1 | item), VerbAgg, binomial)

## Fixed persons, random items
m7 <- glmer(r2 ~ 0 + id + (1 | item), VerbAgg, binomial)
```

Example: Verbal Aggression

Person covariate models

- Latent regression 1PL model (person property covariates)

```
m8 <- glmer(r2 ~ 0 + item + Anger + Gender + (1 | id), VerbAgg, binomial)

## Heteroscedasticity on gender
m9 <- glmer(r2 ~ 0 + item + Anger + Gender
               + (0 + Gender | id), VerbAgg, binomial)
```

- Multilevel model (person partition covariates)

```
m10 <- glmer(r2 ~ 0 + item + (1 | id) + (1 | group), VerbAgg, binomial)
```

Example: Verbal Aggression

Person-by-item covariates

- Differential item functioning (DIF)

```
## DIF w.r.t. gender for 8 items
dif <- with(VerbAgg,
  factor(0 + (Gender == "F" & mode == "do" & btype != "shout")))

m11  <- glmer(r2 ~ 0 + item + dif + Gender + (1 | id), VerbAgg, binomial)
m11a <- glmer(r2 ~ 0 + item +       Gender + (1 | id), VerbAgg, binomial)
anova(m11, m11a)

## Gender DIF random across persons
m12 <- glmer(r2 ~ 0 + item + dif + Gender + diag(1 + dif | id),
             VerbAgg, binomial)
```

References

De Boeck, P., Bakker, M., Zwitser, R., Nivard, M., Hofman, A., Tuerlinckx, F., & Partchev, I. (2011). The Estimation of Item Response Models with the lmer Function from the lme4 Package in R. *Journal of Statistical Software*, 39(12), 1–28. https://doi.org/10.18637/jss.v039.i12

Doran, H., Bates, D., Bliese, P., & Dowling, M. . (2007). Estimating the Multilevel Rasch Model: With the lme4 Package. *Journal of Statistical Software*, 20(2), 1–18. https://doi.org/10.18637/jss.v020.i02

Rijmen, F., Tuerlinckx, F., De Boeck, P., & Kuppens, P. (2003). A nonlinear mixed model framework for item response theory. *Psychological Methods*, 8(2), 185–205. https://doi.org/10.1037/1082-989X.8.2.185