# Statistik III

# Advanced Statistics

Florian Wickelmaier

EBERHARD KARLS
UNIVERSITÄT
TÜBINGEN



Summer 2026

# Multivariate Tests

Introduction

- Often an independent variable exerts effects on *multiple* dependent variables
- Multivariate analyses are often to be preferred to multiple univariate tests
  - Application of multiple univariate tests leads to Type I error inflation (cumulation of $\alpha$ error)
  - Univariate tests ignore dependencies between variables
  - Sometimes differences cannot be determined reliably on each of various variables, but by simultaneously considering them

# From Univariate to Multivariate Statistics

- As an example, we consider the multivariate analogue of the test of the expected value of a normally distributed random variable with unknown variance
- Univariate version of the one-sample $t$-test
  - For the random variable $X$, assumed to be normally distributed with expected value $\mu$ and unknown variance $\sigma^2$, we want to test the hypotheses

$$H_0 \colon \mu = \mu_0 \quad \text{against} \quad H_1 \colon \mu \neq \mu_0$$

  - This test uses the $t_{(n-1)}$ distributed statistic

$$T = \sqrt{n}\,\frac{\bar{X} - \mu_0}{S}$$

with sample mean $\bar{X}$ and standard deviation $S$ (providing estimates of $\mu$ and $\sigma$)

# From Univariate to Multivariate Statistics

- Univariate version of the one-sample $t$-test
  - The square of this statistic
    $$T^2 = n \, \frac{(\bar{X} - \mu_0)^2}{S^2}$$
    is $F(1, n-1)$-distributed (see Statistics I)
  - We can rearrange this equation as follows
    $$T^2 = n \cdot (\bar{X} - \mu_0) \cdot (S^2)^{-1} \cdot (\bar{X} - \mu_0)$$
- We will see that this equation can be readily generalized to cover the corresponding multivariate case
- The resulting test statistic is called *Hotelling's* $T^2$

# Multivariate One-Sample t-Test

Formulation of the multivariate test

- In the multivariate version of the test, we consider random variables $X_1, \ldots, X_p$ with the following assumptions
  - The random vector $\mathbf{x}' = (X_1, \ldots, X_p)$ follows a multivariate normal distribution with expected value $\boldsymbol{\mu}$ and unknown covariance matrix $\boldsymbol{\Sigma}$
  - By $\bar{\mathbf{x}}$ we denote the mean vector, calculated from $n$ realizations of the random vector $\mathbf{x}$, and with $\mathbf{S}$ we denote the respective sample covariance matrix
- Generalizing the univariate case, we test the hypotheses

$$H_0 \colon \boldsymbol{\mu} = \boldsymbol{\mu}_0 \quad \text{against} \quad H_1 \colon \boldsymbol{\mu} \neq \boldsymbol{\mu}_0$$

# Multivariate One-Sample t-Test

- As in the univariate case we consider the test statistic

$$T^2 = n \cdot (\bar{\mathbf{x}} - \boldsymbol{\mu_0})' \cdot \mathbf{S}^{-1} \cdot (\bar{\mathbf{x}} - \boldsymbol{\mu_0})$$

- Under to the null hypothesis, the statistic

$$F = \frac{n-p}{(n-1)p} T^2$$

  is $F(p, n-p)$-distributed ($p$ and $n-p$ degrees of freedom)
- For the 0.95 confidence region of the expected value $\boldsymbol{\mu}$ we consider the inequality

$$F \leq F_{0.95}(p, n-p)$$

  with $F_{0.95}(p, n-p)$ the 0.95-quantile of the $F(p, n-p)$-distribution

# Multivariate One-Sample t-Test

- Thus, for the confidence region we have

$$\frac{n-p}{(n-1)p}\, n \cdot (\bar{\mathbf{x}} - \boldsymbol{\mu})' \cdot \mathbf{S}^{-1} \cdot (\bar{\mathbf{x}} - \boldsymbol{\mu}) \leq F_{0.95}(p, n-p)$$

- and

$$(\bar{\mathbf{x}} - \boldsymbol{\mu})' \cdot \mathbf{S}^{-1} \cdot (\bar{\mathbf{x}} - \boldsymbol{\mu}) \leq \frac{(n-1)p}{(n-p)n}\, F_{0.95}(p, n-p)$$

- This means that the expected value is within the 0.95 confidence region if its Mahalanobis distance to the mean vector is not greater than

$$\sqrt{\frac{(n-1)p}{(n-p)n}\, F_{0.95}(p, n-p)}$$

# Example: Multivariate One-Sample t-Test

Two univariate tests

- Data from $n = 37$ participants taking the subtests "Similarities" and "Arithmetic" of the Wechsler Adult Intelligence Scale (WAIS)
- After normalization we expect a score of 10 on both scales
- Univariate tests of the subtest-specific null hypotheses $H_0 \colon \mu = 10$

| | $\bar{x}$ | $s$ | $t(36)$ | $p$ |
|---|---|---|---|---|
| Similarities | 9.568 | 3.476 | -0.757 | 0.454 |
| Arithmetic | 11.486 | 3.330 | 2.715 | 0.010 |

# Example: Multivariate One-Sample t-Test

Two univariate tests

- The univariate two-sided 0.95 confidence interval for the subtest "Similarities" is

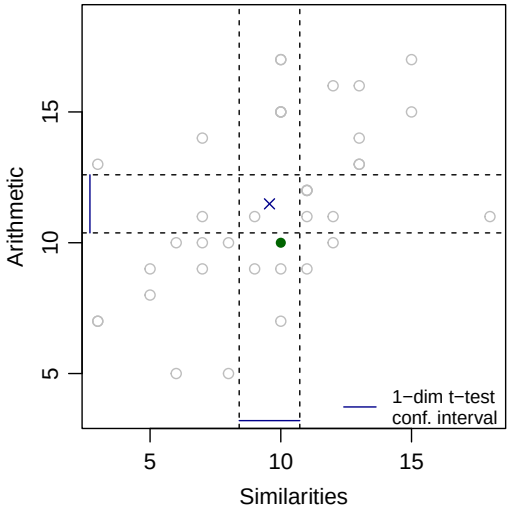
$$\begin{aligned}
&\left[\bar{x} - t_{0.975}(n-1) \cdot \frac{s}{\sqrt{n}},\ \bar{x} + t_{0.975}(n-1) \cdot \frac{s}{\sqrt{n}}\right] \\
&= \left[9.568 - 2.028 \cdot \frac{3.476}{6.083},\ 9.568 + 2.028 \cdot \frac{3.476}{6.083}\right] \\
&= [8.408, 10.727]
\end{aligned}$$

and for the subtest "Arithmetic"

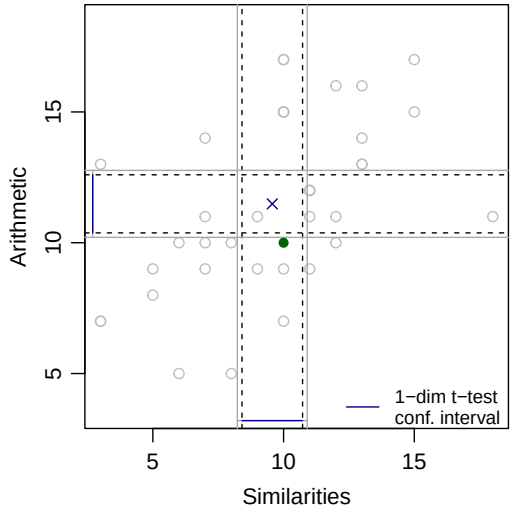
$$\begin{aligned}
&\left[11.486 - 2.028 \cdot \frac{3.330}{6.083},\ 11.486 + 2.028 \cdot \frac{3.330}{6.083}\right] \\
&= [10.376, 12.597]
\end{aligned}$$

# Example: Multivariate One-Sample t-Test

Two univariate tests with $\bar{\mathbf{x}}$ and $\boldsymbol{\mu_0}$




Bonferroni-adjusted CIs added



# Example: Multivariate One-Sample t-Test

Multivariate test

- In the multivariate case we consider

$$\mathbf{S} = \begin{pmatrix} 12.086 & 5.938 \\ 5.938 & 11.090 \end{pmatrix}$$

and

$$\mathbf{S}^{-1} = \begin{pmatrix} 0.112 & -0.060 \\ -0.060 & 0.122 \end{pmatrix}$$

- Under the assumption of a joint multivariate normal distribution, we test the null hypothesis

$$H_0 : \boldsymbol{\mu} = \boldsymbol{\mu}_0 = \begin{pmatrix} 10 \\ 10 \end{pmatrix}$$

# Example: Multivariate One-Sample t-Test

Multivariate test

- We get

$$
\begin{aligned}
T^2 &= n \cdot (\bar{\mathbf{x}} - \boldsymbol{\mu_0})' \cdot \mathbf{S}^{-1} \cdot (\bar{\mathbf{x}} - \boldsymbol{\mu_0}) \\
T^2 &= 37 \cdot (-0.432, 1.486) \cdot \begin{pmatrix} 0.112 & -0.060 \\ -0.060 & 0.122 \end{pmatrix} \cdot \begin{pmatrix} -0.432 \\ 1.486 \end{pmatrix} \\
&= 13.641
\end{aligned}
$$

- and the test statistic

$$
F = \frac{n-p}{(n-1)p} \cdot T^2 = \frac{37-2}{(37-1)2} \cdot 13.641 = 6.631
$$

is $F(2, 35)$-distributed with $p$-value $p < 0.004$

# Example: Multivariate One-Sample t-Test

Multivariate test

- With

$$(\boldsymbol{\mu} - \bar{\mathbf{x}})' \cdot \mathbf{S}^{-1} \cdot (\boldsymbol{\mu} - \bar{\mathbf{x}}) \leq \frac{(n-1)p}{(n-p)n} F_{0.95}(p, n-p)$$

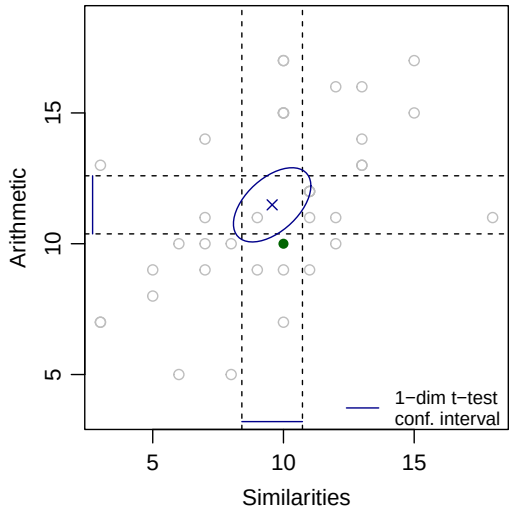
the 0.95 confidence region for the expected value $\boldsymbol{\mu}$ is the ellipse defined by the Mahalanobis distance

$$\sqrt{\frac{(n-1)p}{(n-p)n} F_{0.95}(p, n-p)} = \sqrt{\frac{(37-1)2}{(37-2)37} 3.267} = 0.426$$

from the mean vector $\bar{\mathbf{x}}$

# Example: Multivariate One-Sample t-Test

Multivariate test with $\bar{\mathbf{x}}$ and $\boldsymbol{\mu_0}$



# Multivariate One-Sample t-Test in R

Example

- Data from $n = 37$ participants taking the subtests "Information," "Similarities," "Arithmetic," and "Picture Completion" of the Wechsler Adult Intelligence Scale (WAIS)

```
dat <- read.table("wechsler.txt", header = TRUE)  # remember to setwd()
mu0 <- rep(10, 4)

X <- subset(dat, select = -ID) |> as.matrix()     # without ID
n <- nrow(X)
p <- ncol(X)
x_bar <- colMeans(X)
S <- cov(X)
```

# Multivariate One-Sample t-Test in R

Example

```
T2 <- n * t(x_bar - mu0) %*% solve(S) %*% (x_bar - mu0)
F <- (n - p) / (p * (n - 1)) * T2
p.value <- 1 - pf(F, p, n - p)
qF <- qf(.95, p, n - p)           # critical quantile

m1 <- lm(X ~ 1)                   # multivariate model
confint(m1, level = 1 - .05/p)    # Bonferroni adjusted CIs
#          Info  Sim Arith Pic
#  0.625 % 11.1  8.1  10.0 7.1
# 99.375 % 14.0 11.1  12.9 8.8

m2 <- lm(X ~ offset(rep(10, n)))  # H0: mu = 10
summary(m2)                       # univariate t tests
anova(m2)                         # multivariate test
```

# Principles of Test Construction

Union-Intersection Principle

- In principle, multivariate tests can also be thought of as multiple univariate tests
- Therefore, the question arises if Type I error inflation is still a problem
- That this is not the case follows from a fundamental principle of multivariate statistics, which is the

  *Union-Intersection Principle (UI Principle)*

- The following explanation of its implementation for Hotelling's $T^2$ in the one-sample case will clarify the question of possible Type I error inflation

# Union-Intersection Principle

- In addition to the complex multivariate null hypothesis $H_0$, we formulate an appropriate set of simple, in this case univariate, null hypotheses $H_{0i}$ ($i$ from an appropriate index set $I$), the joint validity of which is equivalent to the validity of $H_0$

  *This corresponds to the intersection of events "$H_{0i}$ is valid"*

- Next we test all $H_{0i}$ (where $\alpha$ needs to be adjusted) and reject $H_0$ if and only if at least one $H_{0i}$ is rejected

  *$H_0$ is rejected if and only if the union of the events "reject $H_{0i}$" occurs*

# UI Principle: Hotelling's $T^2$

Univariate hypotheses $H_{0\mathbf{c}}$

- Consider the $N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$-distributed random vector $\mathbf{x}$
- With a $p$-vector $\mathbf{c} \neq \mathbf{0}$ we can assign a one-dimensional $N(\mathbf{c}'\boldsymbol{\mu}, \mathbf{c}'\boldsymbol{\Sigma}\mathbf{c})$-distributed random variable to the random vector $\mathbf{x}$, by means of the scalar product $\mathbf{c}'\mathbf{x}$
- If the null hypothesis

  $$H_0 \colon \boldsymbol{\mu} = \boldsymbol{\mu_0}$$

  is valid, then the null hypotheses $H_{0\mathbf{c}}$ for all of these $\mathbf{c}$ are valid

  $$H_{0\mathbf{c}} \colon \mathbf{c}'\boldsymbol{\mu} = \mathbf{c}'\boldsymbol{\mu_0}$$

- Therefore, the validity of $H_0$ implies the joint validity of all univariate null hypotheses $H_{0\mathbf{c}}$

  $$H_0 \text{ valid} \Rightarrow \text{all } H_{0\mathbf{c}} \text{ valid}$$

# UI Principle: Hotelling's $T^2$

Univariate hypotheses $H_{0\mathbf{c}}$

- If $H_0$ *does not hold true*, meaning

$$\boldsymbol{\mu} \neq \boldsymbol{\mu_0},$$

then there is a component $k \in \{1, \ldots, p\}$ with $\mu_k \neq \mu_{0_k}$

- It follows that for $\mathbf{c}' = (c_1, \ldots, c_j, \ldots, c_p)$ defined by

$$c_j = \begin{cases} 1 & \text{for } j = k \\ 0 & \text{else} \end{cases}$$

that the null hypotheses $H_{0\mathbf{c}}$ do not hold true, meaning

$$\mathbf{c}'\boldsymbol{\mu} \neq \mathbf{c}'\boldsymbol{\mu_0}$$

- Therefore, the joint validity of all univariate null hypotheses $H_{0\mathbf{c}}$ is equivalent to the validity of $H_0$

$$H_0 \text{ valid} \Rightarrow \text{all } H_{0\mathbf{c}} \text{ valid}, \quad H_0 \text{ not valid} \Rightarrow \text{not all } H_{0\mathbf{c}} \text{ valid}$$

# UI Principle: Hotelling's $T^2$

- The null hypotheses $H_{0\mathbf{c}}$ are to be tested via the $t$ statistics (two-sided)

$$t = \sqrt{n}\,\frac{\mathbf{c}'\bar{\mathbf{x}} - \mathbf{c}'\boldsymbol{\mu_0}}{\sqrt{\mathbf{c}'\mathbf{S}\mathbf{c}}}$$

  where the significance level needs to be adjusted in order to determine the critical values
- The adjusted significance level $\tilde{\alpha}$ needs to assure that for a valid $H_0$ the probability of rejecting any of the $H_{0\mathbf{c}}$ is less than or equal to the originally chosen $\alpha$

# UI Principle: Hotelling's $T^2$

- We need not set the adjusted significance level $\tilde{\alpha}$ explicitly, because it can be shown that the squared $t$-values

$$n \frac{(\mathbf{c}'\bar{\mathbf{x}} - \mathbf{c}'\boldsymbol{\mu_0})^2}{\mathbf{c}'\mathbf{S}\mathbf{c}}$$

  attain a maximum, which is Hotelling's $T^2$

$$n\,(\bar{\mathbf{x}} - \boldsymbol{\mu_0})'\mathbf{S}^{-1}(\bar{\mathbf{x}} - \boldsymbol{\mu_0})$$

- Therefore, at least one $H_{0\mathbf{c}}$ will be rejected if $T^2$ becomes significant with the given significance level $\alpha$
- With this decision all univariate null hypotheses $H_{0\mathbf{c}}$ will be tested simultaneously without explicitly setting an adjusted $\tilde{\alpha}$

# Example: Union-Intersection Principle

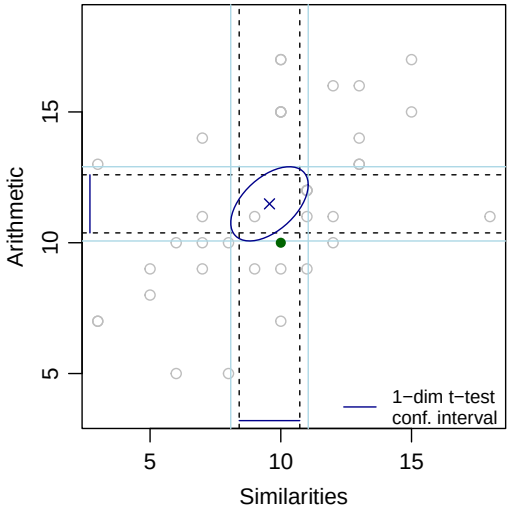
- The univariate tests of the WAIS data subtests "Similarities" and "Arithmetic" results by using the scalar product with

$$\mathbf{c} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{c} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

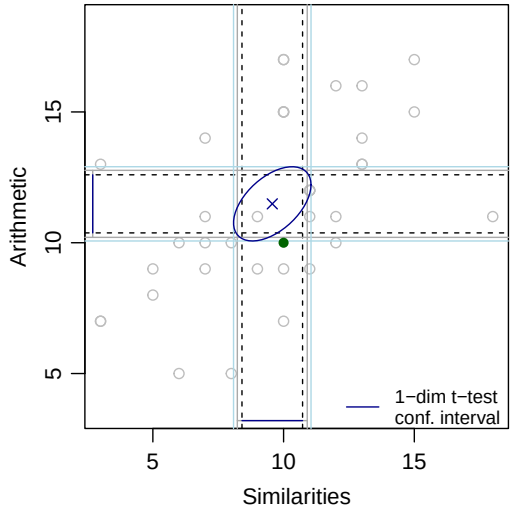
  which can be interpreted as one-dimensional projections of the data points onto the coordinate axes
- The deviation of the one-dimensional projections of the confidence ellipse onto the coordinate axes from the univariate confidence intervals is due to the missing $\alpha$ adjustments in the univariate cases
- From a multivariate perspective, Bonferroni adjustment of $\alpha$ for the $p = 2$ univariate tests is insufficient

# Example: Union-Intersection Principle

Multi- and univariate tests with $\bar{\mathbf{x}}$ and $\boldsymbol{\mu_0}$




Bonferroni-adjusted CIs added

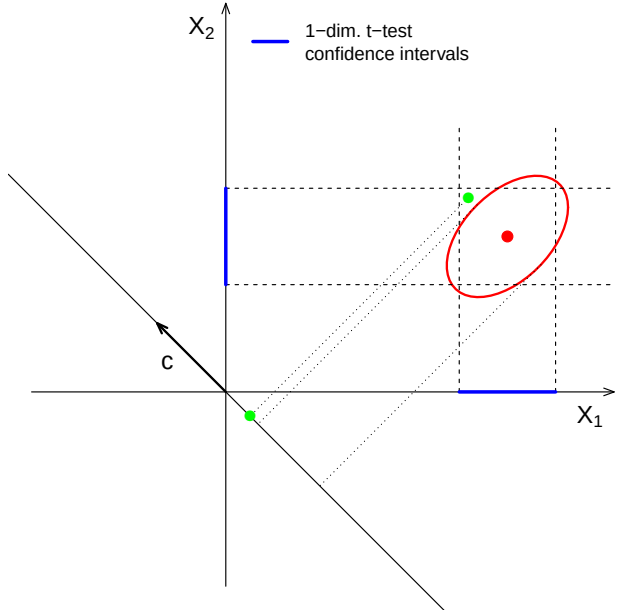


# Union-Intersection Principle

- In general a significant result of the multivariate test need not imply a significant results of any one of the univariate tests on the given variables (in contrast to the presented example)
- The union-intersection principle implies that there is a coefficient vector $\mathbf{c} \neq \mathbf{0}$ such that the univariate test of $H_{0\mathbf{c}}$ is significant
- The vector $\mathbf{c}$, however, does not have to be a column vector of the respective identity matrix $\mathbf{I}_p$, representing a one-dimensional projection onto one of the coordinate axes
- In the following example of multivariate significance
  - the univariate test for the chosen vector $\mathbf{c}$ is significant
  - none of the univariate tests for any of coordinates are significant

# Example: Union-Intersection Principle

Multi- and univariate tests



# Two-Sample Case

In the following we will consider methods to compare multivariate expected values of two samples

As in univariate statistics, in multivariate statistics we distinguish two cases

- Independent samples
  - Multivariate version of the univariate $t$-test for independent samples
- Dependent samples (paired observations)
  - Multivariate version of the univariate $t$-test for dependent samples

# Multivariate Independent-Sample t-Test

Indication

- We use *Hotelling's $T^2$ for independent samples* if we test mean differences of two groups in $p$ variables which can be assumed to be independent across groups
- Independent groups (samples) arise for example from
  - Randomization in experimental designs (e. g., treatment vs. control group)
  - Division in "natural groups" in the course of quasi-experimental designs (e. g., female vs. male)
- Covariances matrices of the two groups are unknown, but satisfy certain properties specified below

# Example: Coronary Heart Disease

- In a study of risk factors of coronary heart disease (CHD) at the beginning of the study period age ($X_1$), blood pressure ($X_2$), and cholesterol level ($X_3$) were determined for a sample of $n_1 = 832$ healthy male participants
- At the end of the study period $n_2 = 71$ of the participants suffered from a coronary heart disease
- The following means were obtained

| | healthy ($n_1 = 761$) | CHD ($n_2 = 71$) |
|---|---|---|
| $X_1$ | 44.81 | 56.86 |
| $X_2$ | 86.99 | 95.62 |
| $X_3$ | 201.27 | 221.51 |

# Multivariate Independent-Sample t-Test

Assumptions

- The sample
$$\mathbf{X}_1 = (\mathbf{x}_1^{(1)}, \ldots, \mathbf{x}_{n_1}^{(1)})'$$
comes from a $N_p(\boldsymbol{\mu}_1, \boldsymbol{\Sigma})$-distributed population
- The sample
$$\mathbf{X}_2 = (\mathbf{x}_1^{(2)}, \ldots, \mathbf{x}_{n_2}^{(2)})'$$
comes from a $N_p(\boldsymbol{\mu}_2, \boldsymbol{\Sigma})$-distributed population
- The samples $\mathbf{X}_1$ and $\mathbf{X}_2$ are stochastically independent, and the respective covariance matrices satisfy
$$Cov(\mathbf{X}_1) = Cov(\mathbf{X}_2) = \boldsymbol{\Sigma}$$

# Multivariate Independent-Sample t-Test

Remarks

- Both samples are assumed to be independent and multivariate normally distributed with equal covariance matrices
- Notice that this assumption is much stricter than the assumption equal variances in univariate statistics
- Samples only differ with respect to their mean vectors

Hypotheses

- Null hypothesis

$$H_0 \colon \boldsymbol{\mu}_1 = \boldsymbol{\mu}_2$$

- Alternative hypothesis

$$H_1 \colon \boldsymbol{\mu}_1 \neq \boldsymbol{\mu}_2$$

# Multivariate Independent-Sample t-Test

From univariate to multivariate statistics

- The test statistic of the *univariate* $t$-test for independent samples is

$$t = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\left(\frac{1}{n_1} + \frac{1}{n_2}\right) \frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}}} = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\left(\frac{n_1 + n_2}{n_1 n_2}\right) s^2}}$$

  with $s^2$ the pooled sample variance an unbiased estimator of $\sigma^2$

- Taking squares we obtain the $F(1, n_1 + n_2 - 2)$-distributed statistics

$$t^2 = \frac{n_1 n_2}{n_1 + n_2} (\bar{x}_1 - \bar{x}_2)(s^2)^{-1}(\bar{x}_1 - \bar{x}_2)$$

# Multivariate Independent-Sample t-Test

Test statistic

- Analogously, in the multivariate case we compute

$$T^2 = \frac{n_1 n_2}{n_1 + n_2}(\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2)' \mathbf{S}^{-1} (\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2)$$

where $\bar{\mathbf{x}}_1$ and $\bar{\mathbf{x}}_2$ are the mean vectors of the two groups and $\mathbf{S}$ is the pooled covariance matrix

$$\mathbf{S} = \frac{1}{n_1 + n_2 - 2}[(n_1 - 1)\mathbf{S}_1 + (n_2 - 1)\mathbf{S}_2]$$

with $\mathbf{S}_1$ and $\mathbf{S}_2$ the within-groups covariance matrices

# Multivariate Independent-Sample t-Test

Decision

- Under the null hypothesis the test statistic

$$F = \frac{n_1 + n_2 - p - 1}{(n_1 + n_2 - 2) \cdot p} \cdot T^2$$

  is $F(p, n_1 + n_2 - p - 1)$-distributed
- For a given significance level $\alpha$ the null hypothesis will be rejected whenever

$$F > F_{1-\alpha}(p, n_1 + n_2 - p - 1)$$

# Example: Multivariate Independent-Sample t-Test

- Because the presence of CHD was observed independently of the $p = 3$ variables, two subsamples of $n_1 = 761$ healthy and $n_2 = 71$ CHD participants were formed
- The following mean vectors were obtained

$$\bar{\mathbf{x}}_1 = \begin{pmatrix} 44.81 \\ 86.99 \\ 201.27 \end{pmatrix}, \ \bar{\mathbf{x}}_2 = \begin{pmatrix} 56.86 \\ 95.62 \\ 221.51 \end{pmatrix} \text{ and } \bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2 = \begin{pmatrix} -12.05 \\ -8.63 \\ -20.24 \end{pmatrix}$$

- The pooled covariance matrix is

$$\mathbf{S} = \begin{pmatrix} 214.26 & 72.73 & 195.61 \\ 72.73 & 212.44 & 175.53 \\ 195.61 & 175.53 & 1820.61 \end{pmatrix}$$

# Example: Multivariate Independent-Sample t-Test

- This provides the correlation matrix

$$\mathbf{R} = \begin{pmatrix} 1 & 0.341 & 0.313 \\ 0.341 & 1 & 0.282 \\ 0.313 & 0.282 & 1 \end{pmatrix}$$

(e. g., $r_{12} = S_{12}/\sqrt{S_{11}S_{22}} = 72.73/\sqrt{214.26 \cdot 212.44} = 0.341$)

- The inverse of the covariance matrix is

$$\mathbf{S}^{-1} = \frac{1}{63503292} \begin{pmatrix} 355959.61 & -98077.54 & -28789.09 \\ -98077.54 & 351820.63 & -23382.34 \\ -28789.09 & -23382.34 & 40227.74 \end{pmatrix}$$

# Example: Multivariate Independent-Sample t-Test

- For the test statistic we compute

$$\begin{aligned} T^2 &= \frac{n_1 n_2}{n_1 + n_2}(\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2)' \mathbf{S}^{-1} (\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2) \\ &= \frac{761 \cdot 71}{832} \begin{pmatrix} -12.05 & -8.63 & -20.24 \end{pmatrix} \mathbf{S}^{-1} \begin{pmatrix} -12.05 \\ -8.63 \\ -20.24 \end{pmatrix} \\ &= 52.93 \end{aligned}$$

  and

$$F = \frac{n_1 + n_2 - p - 1}{(n_1 + n_2 - 2) \cdot p} \cdot T^2 = \frac{761 + 71 - 3 - 1}{(761 + 71 - 2) \cdot 3} \cdot T^2 = 17.60$$

- As $F_{0.95}(3, 828) = 2.62$, the test statistic falls into the rejection region and the groups differ significantly with respect to the variables

# Multivariate Independent-Sample t-Test in R

Designing a confirmatory study: sample size for 85% power

```
n1 <- n2 <- 60              # sample size per group
sds <- c(15, 15, 44)        # standard deviations
r <- 0.3                    # correlation, here constant

S <- r * sds %*% t(sds)
diag(S) <- sds^2
dat <- data.frame(grp = rep(c("trt", "ctl"), c(n1, n2)))

pval <- replicate(2000, {  # repetitions of the experiment
  dat$y <- rbind(
    MASS::mvrnorm(n1, mu = c(45 + 10, 85 + 5, 200 + 15), Sigma = S),
    MASS::mvrnorm(n2, mu = c(45     , 85    , 200     ), Sigma = S)
  )
  anova(lm(y ~ grp, dat))$"Pr(>F)"[2]  # p-value of the test
})
mean(pval < 0.05)           # simulated power with alpha = 0.05
```

# Multivariate Paired-Sample t-Test

Indication

- *Hotelling's $T^2$ for dependent samples* is used if we test mean differences of two groups in $p$ variables, but we cannot assume independent samples
- Examples for *paired observations* or *matched pairs*
  - Repeated measures designs, where for each unit (e. g., person) we observe $p$ variables at two time points (= groups)
  - Subjects are matched according to some criterion or control variables (homogeneous grouping)
- The covariance matrices of both time points (groups) again are unknown

# Multivariate Paired-Sample t-Test

Data

- We consider the $p$-vectors
$$\mathbf{d}_i = \mathbf{x}_i^{(1)} - \mathbf{x}_i^{(2)}$$
containing the differences between groups in each of the $p$ variables across all units ($i = 1, \ldots, n$)

Assumptions

- The vectors of differences are stochastically independent and multivariate normally distributed,
$$\mathbf{d}_i \sim N_p(\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2, \boldsymbol{\Sigma}_d)$$

Hypotheses

- Null hypothesis $H_0 \colon \boldsymbol{\mu}_1 = \boldsymbol{\mu}_2 \;\Leftrightarrow\; \boldsymbol{\mu}_1 - \boldsymbol{\mu}_2 = \mathbf{0}$
- Alternative hypothesis $H_1 \colon \boldsymbol{\mu}_1 \neq \boldsymbol{\mu}_2$

# Multivariate Paired-Sample t-Test

From univariate to multivariate statistics

- The test statistic for the *univariate* $t$-test for dependent samples is

$$t = \frac{\bar{d}}{s_d}\sqrt{n}$$

  with the mean difference $\bar{d}$ and the standard deviation $s_d$ of the differences

- Taking squares leads to

$$t^2 = n \cdot \bar{d} \cdot (s_d^2)^{-1} \cdot \bar{d}$$

# Multivariate Paired-Sample t-Test

Test statistic

- In analogy to the already considered cases, *Hotelling's $T^2$ for dependent samples* is computed by

$$T^2 = n \cdot \bar{\mathbf{d}}' \cdot \mathbf{S}_d^{-1} \cdot \bar{\mathbf{d}}$$

with mean difference vector

$$\bar{\mathbf{d}} = \frac{1}{n} \sum_{i=1}^{n} \mathbf{d}_i$$

and covariance matrix of the difference vectors

$$\mathbf{S}_d = \frac{1}{n-1} \sum_{i=1}^{n} (\mathbf{d}_i - \bar{\mathbf{d}})(\mathbf{d}_i - \bar{\mathbf{d}})'$$

# Multivariate Paired-Sample t-Test

Decision

- Under the null hypothesis

$$F = \frac{n-p}{(n-1)\cdot p} \cdot T^2$$

  is $F(p, n-p)$-distributed
- For given $\alpha$ level, the null hypothesis is rejected if

$$F > F_{1-\alpha}(p, n-p)$$

# Multivariate Paired-Sample t-Test in R

Example (Bortz & Schuster, 2010, p. 474f.)

- Eight persons rate their social anxiety and their urge to dominate before and after a group dynamic training (7-point Likert scale)

```
dat <- read.table(header = TRUE, text = "
  AnxPre DomPre AnxPost DomPost
       5      3       3       3
       4      3       3       4
       6      2       2       3
       6      3       4       4
       7      2       5       4
       5      4       3       3
       4      4       2       4
       3      3       2       5
")
```

# Multivariate Paired-Sample t-Test in R

Example (Bortz & Schuster, 2010, p. 474f.)

```
d <- cbind(Anx = dat[, "AnxPre"] - dat[, "AnxPost"],  # change scores
           Dom = dat[, "DomPre"] - dat[, "DomPost"])
d_bar <- colMeans(d)                                  # centroid
S <- cov(d)

m <- lm(d ~ 1)                                        # multivariate model
summary(m)                                            # univariate tests
anova(m)    # F(2, 6) = 19.84, p = .002               # multivariate test
```

# Generalization

Multivariate analysis of variance (MANOVA)

- As in univariate statistics, we have the following generalizations of statistical tests based on Hotelling's $T^2$
  - For more than two *independent* groups, the approach may be generalized to *MANOVA*
  - For more than two *dependent* groups, the approach may be generalized to *MANOVA with repeated measures*
- These procedures can be conceived as special cases of the *general linear model*

# References

Bortz, J. & Schuster, C. (2010). Statistik für Sozialwissenschaftler. Berlin: Springer. (Kapitel 26)

Fahrmeir, L., Hamerle, A. & Tutz, G. (1996). Multivariate statistische Verfahren. 2. Auflage. Berlin: De Gruyter. (Kapitel 3)

Stevens, J. (2001). Applied multivariate statistics for the social sciences. Hillsdale, NJ: Lawrence Erlbaum Associates. (Chapter 4)