Statistik III

Advanced Statistics

Florian Wickelmaier

EBERHARD KARLS
UNIVERSITÄT
TÜBINGEN



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Basic Concepts of Multivariate Statistics

Multivariate random variables

- Basic concepts of linear algebra will be applied to multivariate statistics
- We will look at multivariate generalizations of concepts from
  - descriptive statistics
  - probability theory
- Grounded on these notions, we consider
  - multivariate versions of procedures known from univariate statistics
  - new multivariate methods

Data Matrix

Data basis

- As a starting point for descriptive statistics we, will use the $(n \times p)$-matrix $\mathbf{X}$ (*data matrix*) consisting of p realizations of variables $X_1, \ldots, X_p$ for each person from a sample of size n

$$\mathbf{X} = \begin{pmatrix} x_{11} & x_{12} & \ldots & x_{1p} \\ x_{21} & x_{22} & \ldots & x_{2p} \\ \vdots & \vdots & & \vdots \\ x_{n1} & x_{n2} & \ldots & x_{np} \end{pmatrix}$$

- The $(n \times p)$ data matrix $\mathbf{X}$ is composed of the p column vectors $\mathbf{x}_1, \ldots, \mathbf{x}_p$, each containing the values of one variable across all n persons

$$\mathbf{X} = \begin{pmatrix} \mathbf{x}_1, & \ldots, & \mathbf{x}_p \end{pmatrix}$$

Mean Vector

Definition and calculation

- Analogous to univariate statistics we consider the *mean vector* $\bar{\mathbf{x}}$ (sometimes called *centroid*)
  - composed of the means of each variable computed across persons
- Starting from the $(n \times p)$ data matrix $\mathbf{X}$, the mean vector $\bar{\mathbf{x}}$ is calculated by

$$\bar{\mathbf{x}} = \frac{1}{n} \begin{pmatrix} \sum_{i=1}^{n} x_{i1} \\ \sum_{i=1}^{n} x_{i2} \\ \vdots \\ \sum_{i=1}^{n} x_{ip} \end{pmatrix} = \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \vdots \\ \bar{x}_p \end{pmatrix}$$

Mean Vector

Calculation

- In matrix notation, the mean vector is obtained by

$$\bar{\mathbf{x}} = \frac{1}{n}\mathbf{X}'\mathbf{1}$$

where $\mathbf{1}$ is the n-column vector containing ones only (vector of 1s)

Example

$$\mathbf{X} = \begin{pmatrix} 1 & 1 \\ 3 & 4 \\ 2 & 7 \end{pmatrix}$$

$$\bar{\mathbf{x}} = \frac{1}{3}\begin{pmatrix} 1 & 3 & 2 \\ 1 & 4 & 7 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{3}(1+3+2) \\ \frac{1}{3}(1+4+7) \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

Covariance Matrix

Concept

- Analogous to empirical variance in univariate statistics, we consider and calculate the *(sample) covariance matrix* **S**
- The covariance matrix is a square $(p \times p)$-matrix and contains the variances of the variables (on the diagonal) and their pairwise covariances
- As opposed to the univariate case, appropriately describing the variability of a data set requires to consider variances *and* covariances of the involved variables, which are collected in the covariance matrix **S**

Covariance Matrix

Calculation

- We first compute the $(n \times p)$-matrix $\bar{\mathbf{X}}$, containing the transposed mean vector $\bar{\mathbf{x}}$ in each of the n rows

$$
\begin{aligned}
\bar{\mathbf{X}} &= \frac{1}{n} \mathbf{1}\mathbf{1}'\mathbf{X} \\
&= \frac{1}{n} \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \dots & 1 \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} & \dots & x_{1p} \\ x_{21} & x_{22} & \dots & x_{2p} \\ \vdots & \vdots & & \vdots \\ x_{n1} & x_{n2} & \dots & x_{np} \end{pmatrix} \\
&= \begin{pmatrix} \bar{x}_1 & \bar{x}_2 & \dots & \bar{x}_p \\ \bar{x}_1 & \bar{x}_2 & \dots & \bar{x}_p \\ \vdots & \vdots & & \vdots \\ \bar{x}_1 & \bar{x}_2 & \dots & \bar{x}_p \end{pmatrix}
\end{aligned}
$$

Covariance Matrix

Calculation

- Next, we consider the difference

$$\mathbf{X} - \bar{\mathbf{X}} = \begin{pmatrix} x_{11} & x_{12} & \dots & x_{1p} \\ x_{21} & x_{22} & \dots & x_{2p} \\ \vdots & \vdots & & \vdots \\ x_{n1} & x_{n2} & \dots & x_{np} \end{pmatrix} - \begin{pmatrix} \bar{x}_1 & \bar{x}_2 & \dots & \bar{x}_p \\ \bar{x}_1 & \bar{x}_2 & \dots & \bar{x}_p \\ \vdots & \vdots & & \vdots \\ \bar{x}_1 & \bar{x}_2 & \dots & \bar{x}_p \end{pmatrix}$$

$$= \begin{pmatrix} x_{11} - \bar{x}_1 & x_{12} - \bar{x}_2 & \dots & x_{1p} - \bar{x}_p \\ x_{21} - \bar{x}_1 & x_{22} - \bar{x}_2 & \dots & x_{2p} - \bar{x}_p \\ \vdots & \vdots & & \vdots \\ x_{n1} - \bar{x}_1 & x_{n2} - \bar{x}_2 & \dots & x_{np} - \bar{x}_p \end{pmatrix}$$

Covariance Matrix

Calculation

- To obtain a summation of the squared values in the columns of the matrix of differences $\mathbf{X} - \bar{\mathbf{X}}$, we build the product

$$(\mathbf{X} - \bar{\mathbf{X}})'(\mathbf{X} - \bar{\mathbf{X}})$$

and obtain the $(p \times p)$-matrix (SSCP-matrix, sums of squares and cross products)

$$\begin{pmatrix} \sum_{i=1}^{n}(x_{i1} - \bar{x}_1)^2 & \sum_{i=1}^{n}(x_{i1} - \bar{x}_1) \cdot (x_{i2} - \bar{x}_2) & \dots \\ \sum_{i=1}^{n}(x_{i2} - \bar{x}_2) \cdot (x_{i1} - \bar{x}_1) & \sum_{i=1}^{n}(x_{i2} - \bar{x}_2)^2 & \dots \\ \vdots & \vdots & \end{pmatrix}$$

- The (sample) covariance matrix $\mathbf{S}$ then is given by

$$\mathbf{S} = \frac{1}{n-1}(\mathbf{X} - \bar{\mathbf{X}})'(\mathbf{X} - \bar{\mathbf{X}})$$

Example: Covariance Matrix

- Data matrix and mean vector

$$\mathbf{X} = \begin{pmatrix} 1 & 1 \\ 3 & 4 \\ 2 & 7 \end{pmatrix} \quad \text{and} \quad \bar{\mathbf{x}} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

- Matrix of differences $\mathbf{X} - \bar{\mathbf{X}}$

$$\mathbf{X} - \bar{\mathbf{X}} = \begin{pmatrix} 1 & 1 \\ 3 & 4 \\ 2 & 7 \end{pmatrix} - \begin{pmatrix} 2 & 4 \\ 2 & 4 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} -1 & -3 \\ 1 & 0 \\ 0 & 3 \end{pmatrix}$$

- SSCP matrix

$$(\mathbf{X} - \bar{\mathbf{X}})'(\mathbf{X} - \bar{\mathbf{X}}) = \begin{pmatrix} -1 & 1 & 0 \\ -3 & 0 & 3 \end{pmatrix} \begin{pmatrix} -1 & -3 \\ 1 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 3 & 18 \end{pmatrix}$$

- Covariance matrix $\mathbf{S}$

$$\mathbf{S} = \frac{1}{2} \cdot \begin{pmatrix} 2 & 3 \\ 3 & 18 \end{pmatrix} = \begin{pmatrix} 1 & 1.5 \\ 1.5 & 9 \end{pmatrix}$$

Mahalanobis Distance

Mahalanobis distance (distance of $\mathbf{x}$ to $\bar{\mathbf{x}}$ in units of the standard deviations)

- Analogous to the standardization

$$\frac{x - \bar{x}}{s}$$

with respect to mean and standard deviation (especially for normally distributed variables) in the univariate case, in multivariate statistics we consider

$$(\mathbf{x} - \bar{\mathbf{x}})' \mathbf{S}^{-1} (\mathbf{x} - \bar{\mathbf{x}})$$

- This denotes the squared Mahalanobis distance between the data vector $\mathbf{x}$ and the mean vector $\bar{\mathbf{x}}$

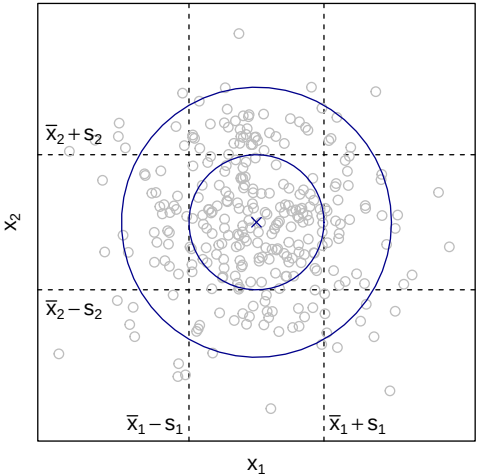
Mahalanobis Distance

- The collection of data vectors $\mathbf{x}$ that have a constant Mahalanobis distance r to the mean vector $\bar{\mathbf{x}}$ are described by the equation

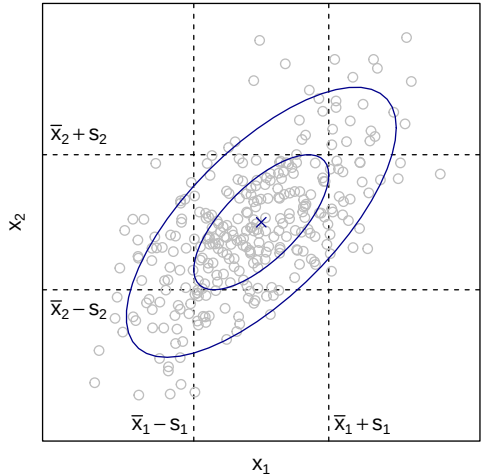
$$(\mathbf{x} - \bar{\mathbf{x}})' \mathbf{S}^{-1} (\mathbf{x} - \bar{\mathbf{x}}) = r^2$$

- Geometrically, these points lie on an ellipsoid around the mean vector $\bar{\mathbf{x}}$, where orientation, shape and size are determined by the covariance matrix and the value of r.
- A characterization of this dependence (based on the notions of eigenvalue and eigenvector) will be provided later
- The following figures illustrate the ellipsoids in the two dimensional case for the respective covariance matrices and $r = 1, 2$

Example: Mahalanobis Distance



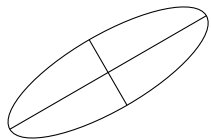
$$\mathbf{S} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$



$$\mathbf{S} = \begin{pmatrix} 1 & 0.7 \\ 0.7 & 1 \end{pmatrix}$$

Mahalanobis Distance

- The points with constant Mahalanobis distance to the centroid form an ellipsoid around the centroid
- In the two dimensional case we obtain an ellipse characterized by the position of its centroid and by the length and orientation of the orthogonal *principal axes*



- Direction and ratio of the lengths of the principle axes are determined by the covariance matrix
- The identification of these parameters pose an *eigenvalue problem*

Eigenvalue Problem

Question

- Can we transform the covariance matrix $\mathbf{S}$ into a diagonal matrix using linear transformations (by preserving relevant characteristics like trace, determinant, ...)?
- This means that we are looking for a matrix $\mathbf{G}$ such that

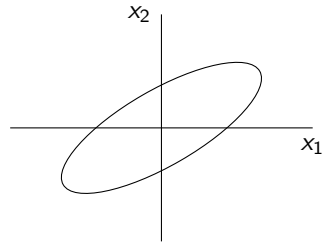
$$\mathbf{G}'\mathbf{S}\mathbf{G} = \mathbf{\Lambda}$$

  where $\mathbf{\Lambda}$ is a diagonal matrix
- The objective of the (coordinate-)transformation is that covariances vanish
- The diagonal elements of $\mathbf{\Lambda}$ then are the eigenvalues of $\mathbf{S}$ and the column vectors of $\mathbf{G}$ their respective eigenvectors

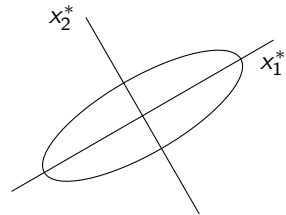
Eigenvalue Problem

Geometrical interpretation

- The data points are correlated when plotted w.r.t. coordinates (x_1, x_2)



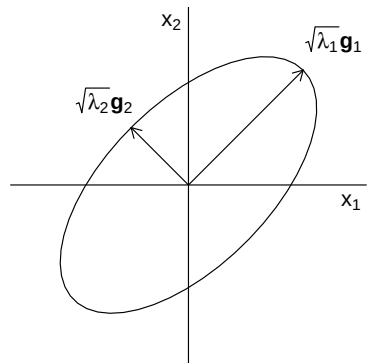
- We are looking for new coordinates (x_1^*, x_2^*) so that in this system the data points are uncorrelated



Eigenvalue Problem

Principal axes of the ellipsoids

- For the principal axes of an ellipsoid determined by a (2×2) covariance matrix $\mathbf{S}$ we have
  - The length of the principal axes is proportional to the square root of the eigenvalues $\sqrt{\lambda_1}$ and $\sqrt{\lambda_2}$ of $\mathbf{S}$ (standard deviation vs. variance)
  - The direction of the principal axes is determined by the respective eigenvectors $\mathbf{g}_1$ and $\mathbf{g}_2$



Total Sample Variance

- The covariance matrix $\mathbf{S}$ provides a complete description of the variances and covariances of the involved variables
- Often we are interested in characterizing the entire variability with a single number (e. g., for statistical tests)
- A first characterization is provided by the sum of the variable variances, called the *total sample variance*
- Because the variances are the diagonal elements of the covariance matrix, the trace may be used for its calculation

$$\mathrm{tr}(\mathbf{S})$$

Total Sample Variance

- With orthogonal coordinate axes and equal units we have the following interpretation
  - The trace of the covariance matrix is the average squared distance of the data points to the centroid (mean vector)
- Under these conditions the trace of the covariance matrix, and thus the total sample variance, can be seen as natural generalization of the univariate variance
- Due to the invariance of the trace of the covariance matrix under its transformation into a diagonal matrix, the total sample variance equals the sum of eigenvalues

Generalized Sample Variance

- The determinant of the covariance matrix

$$\det(\mathbf{S}) = |\mathbf{S}|$$

  provides another characterization of the variability by a single number
- The determinant of the covariance matrix is known as the *generalized sample variance*
- For orthogonal coordinate axes with equal units also the generalized sample variance allows for a geometrical interpretation
  - in the 2-dimensional case as squared area
  - in the higher dimensional case as squared volume

Generalized Sample Variance

- Consider the ellipsoid given by the equation

$$(\mathbf{x} - \bar{\mathbf{x}})' \mathbf{S}^{-1} (\mathbf{x} - \bar{\mathbf{x}}) = 1$$

  of all data points at Mahalanobis distance 1 from the centroid
- Geometrical interpretation of $\det(\mathbf{S})$
  - In two dimensions the determinant equals (up to a dimension-dependent factor) the squared area of the ellipse defined by the above equation
  - In higher dimensions it equals the square of the volume of the respective ellipsoid
- The generalized sample variance of a given covariance matrix equals the product of its eigenvalues

Eigenvalues for the Characterization of Variance

- Other than the total and generalized sample variances, the eigenvalues themselves can be conceived as one-dimensional characterizations of the covariance matrix
  - The eigenvalues determine the lengths of the principal axes (of the ellipsoid)
  - The respective eigenvectors determine their direction
- Therefore, in statistical tests sometimes the greatest eigenvalue of the covariance matrix is used as a proxy to measure the variability of the data

Correlation Matrix

Definition and calculation

- The empirical *correlation matrix*

$$\mathbf{R} = \hat{\mathbf{P}} = (r_{ij})_{p \times p}$$

- is defined by its elements r_{ij} of pairwise correlations

$$r_{ij} = \frac{\sum_{n=1}^{N}(x_{ni} - \bar{x}_i)(x_{nj} - \bar{x}_j)}{\sqrt{\sum_{n=1}^{N}(x_{ni} - \bar{x}_i)^2}\sqrt{\sum_{n=1}^{N}(x_{nj} - \bar{x}_j)^2}}$$

- The correlation can also be obtained from the covariance matrix $\mathbf{S}$ by

$$r_{ij} = \frac{\hat{\sigma}(X_i, X_j)}{\sqrt{\hat{\sigma}^2(X_i)\,\hat{\sigma}^2(X_j)}} = \frac{S_{ij}}{\sqrt{S_{ii}S_{jj}}}$$

  - For $\mathbf{X}$ standardized: $\mathbf{R} = \mathbf{S}$

Descriptive Statistics in R

Data matrix

```
X <- matrix(c(1, 1, 3, 4, 2, 7), 3, 2, byrow = TRUE)
```

Mean vector (centroid) 'by hand'

```
n <- nrow(X)
e <- rep(1, n)
x_bar <- (t(X) %*% e) / n
```

or

```
x_bar <- colMeans(X)
```

Descriptive Statistics in R

Covariance matrix 'by hand'

```
X_bar <- (e %*% t(e) %*% X) / n
SSCP <- t(X - X_bar) %*% (X - X_bar)
S <- SSCP / (n - 1)
```

or

```
S <- cov(X)
```

Correlation matrix

```
R <- cor(X)
R <- cov2cor(S)
```

(Squared) Mahalanobis distance

```
mahalanobis(X, x_bar, S)
```

Descriptive Statistics in R

Eigenvalues and eigenvectors

```
eig <- eigen(S)
lambda <- eig$values
G <- eig$vectors
```

Visualization

```
plot(X[, 1], X[, 2], xlab = expression(x[1]),
     ylab = expression(x[2]), asp = 1)       # data points
car::ellipse(x_bar, S, 1)                    # Mahalanobis distance = 1
arrows(x_bar[1], x_bar[2],                   # principal axes of the ellipse
       x_bar[1] + sqrt(lambda) * G[1, ],
       x_bar[2] + sqrt(lambda) * G[2, ], length = .1)
```

Probability Theory

Introduction

- As in univariate statistics, the introduced notions of descriptive statistics are to be conceived as empirical counterparts of probabilistic concepts
- The mean vector is an estimator for the expected value vector $\boldsymbol{\mu}$, the sample covariance matrix for the (population) covariance matrix $\boldsymbol{\Sigma}$, and the empirical correlation matrix for the (population) correlation matrix $\mathbf{P}$
- We have

$$\begin{aligned} \bar{\mathbf{x}} &= \hat{\boldsymbol{\mu}} \\ \mathbf{S} &= \hat{\boldsymbol{\Sigma}} \\ \mathbf{R} &= \hat{\mathbf{P}} \end{aligned}$$

Expected Value

Definition

- For a given random vector $\mathbf{x} = (X_1, X_2, \ldots, X_p)$ the *expected value (vector)* is defined (element-wise) by

$$\boldsymbol{\mu} = E(\mathbf{x}) = \begin{pmatrix} E(X_1) \\ E(X_2) \\ \vdots \\ E(X_p) \end{pmatrix} = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_p \end{pmatrix}$$

- Accordingly, we can define the expected value of a random matrix $\mathbf{X} = (X_{ij})$ by

$$E(\mathbf{X}) = (E(X_{ij}))$$

Covariance Matrix

Definition

- For a given random vector $\mathbf{x}$ we call

$$\begin{aligned} Cov(\mathbf{x}) &= E[(\mathbf{x}-\boldsymbol{\mu})(\mathbf{x}-\boldsymbol{\mu})'] \\ &= \begin{pmatrix} E(X_1-\mu_1)^2 & \dots & E[(X_1-\mu_1)(X_p-\mu_p)] \\ \vdots & & \vdots \\ E[(X_p-\mu_p)(X_1-\mu_1)] & \dots & E(X_p-\mu_p)^2 \end{pmatrix} \\ &= \begin{pmatrix} \sigma^2(X_1) & \dots & \sigma(X_1,X_p) \\ \vdots & & \vdots \\ \sigma(X_p,X_1) & \dots & \sigma^2(X_p) \end{pmatrix} \end{aligned}$$

  the *covariance matrix* of $\mathbf{x}$, also denoted by $\boldsymbol{\Sigma}$

- $\boldsymbol{\Sigma}$ is symmetrical with variances of $\mathbf{x}$ as diagonal elements and covariances as off-diagonal elements

Expected Values and Covariances

Calculation rules and properties

- Given random vectors $\mathbf{x}$ and $\mathbf{y}$, matrices $\mathbf{A}$ and $\mathbf{B}$, and vectors $\mathbf{a}$, $\mathbf{b}$ (with appropriate dimensions), we have the following calculation rules and properties
  - $E(\mathbf{x}+\mathbf{y}) = E(\mathbf{x}) + E(\mathbf{y})$
  - $E(\mathbf{A}\mathbf{x}+\mathbf{b}) = \mathbf{A}E(\mathbf{x}) + \mathbf{b}$
  - $Cov(\mathbf{x}) = E(\mathbf{x}\mathbf{x}') - \boldsymbol{\mu}\boldsymbol{\mu}'$
  - $Cov(\mathbf{A}\mathbf{x}+\mathbf{b}) = \mathbf{A}Cov(\mathbf{x})\mathbf{A}'$
  - $Cov(\mathbf{x})$ is a symmetric matrix, such that

$$Cov(\mathbf{x}) = Cov(\mathbf{x})'$$

  - $Cov(\mathbf{x})$ is called *positive semidefinite* (*positive definite*) if

$$\mathbf{y}'Cov(\mathbf{x})\mathbf{y} \geq 0 \quad (> 0)$$

    for all $\mathbf{y} \neq \mathbf{0}$

Multivariate Normal Distribution

Definition

- The p-dimensional random vector $\mathbf{x}$ follows a multivariate normal distribution with $E(\mathbf{x}) = \boldsymbol{\mu}$ and $Cov(\mathbf{x}) = \boldsymbol{\Sigma}$, if the density is given by

$$f(\mathbf{x}) = \frac{1}{\sqrt{(2\pi)^p \det(\boldsymbol{\Sigma})}} \exp[-\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{x} - \boldsymbol{\mu})]$$

  where $\boldsymbol{\Sigma}$ is positive definite (then there is an invertible matrix $\mathbf{A}$ such that $\mathbf{A}'\boldsymbol{\Sigma}\mathbf{A}$ is a diagonal matrix with positive entries)

- Analogous to the univariate case we write

$$\mathbf{x} \sim N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$$

Multivariate Normal Distribution

Explanatory notes

- The expression
$$(\mathbf{x} - \boldsymbol{\mu})' \boldsymbol{\Sigma}^{-1} (\mathbf{x} - \boldsymbol{\mu})$$
in the exponent of the exponential function is the squared Mahalanobis distance between $\mathbf{x}$ and the expected value $\boldsymbol{\mu}$
- The *isodensity curves*, which are formed by vectors $\mathbf{x}$ with constant values of the density function (contour lines), are the ellipsoids defined by constant Mahalanobis distance to the expected value $\boldsymbol{\mu}$
- The density is normalized (to guarantee that integration yields 1) by means of the generalized variance $\det(\boldsymbol{\Sigma})$

Example: Multivariate Normal Distribution

- In the 2-dimensional case with

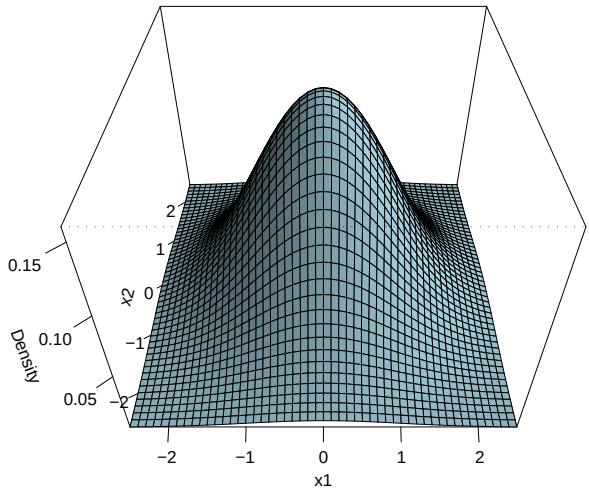
$$\boldsymbol{\mu} = \mathbf{0} \quad \text{and} \quad \boldsymbol{\Sigma} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

the density function for $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ reads

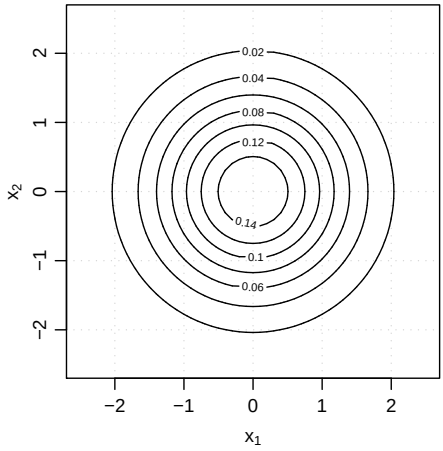
$$f(\mathbf{x}) = \frac{1}{2\pi} \exp[-\frac{1}{2}(x_1^2 + x_2^2)]$$

Example: Multivariate Normal Distribution

Density function for $\boldsymbol{\mu} = \mathbf{0}$ and $\boldsymbol{\Sigma} = \mathbf{I}_2$



Isodensity curves: circles around the origin



Example: Multivariate Normal Distribution

- In the 2-dimensional case with

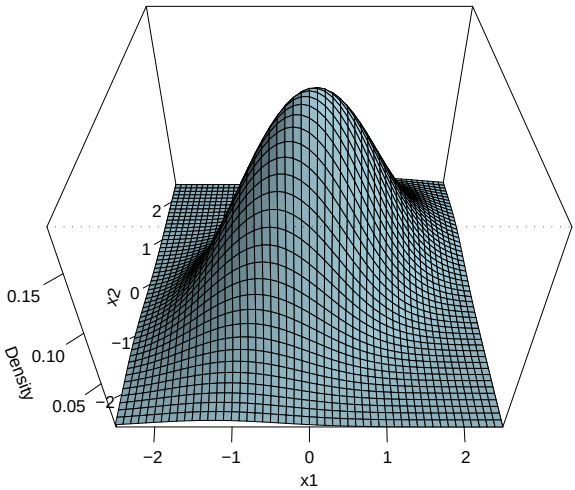
$$\boldsymbol{\mu} = \mathbf{0} \quad \text{and} \quad \boldsymbol{\Sigma} = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$$

the density function for $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ reads

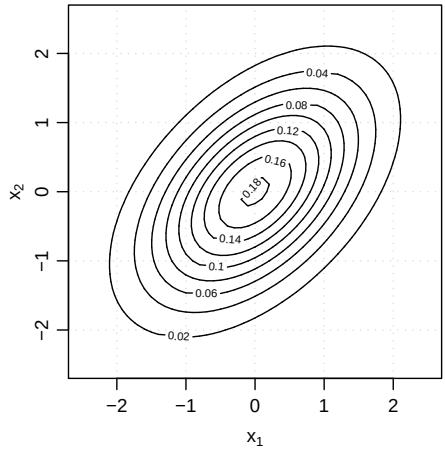
$$f(\mathbf{x}) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp[-\frac{1}{2(1-\rho^2)}(x_1^2 - \rho x_1^2 x_2^2 + x_2^2)]$$

Example: Multivariate Normal Distribution

Density function for $\boldsymbol{\mu} = \mathbf{0}$ and $\rho = 0.5$



Isodensity curves: ellipsoids around the origin



Multivariate Normal Distribution in R

Visualization of the multivariate normal distribution

```
Sigma <- rbind(c(1, .7), c(.7, 1))  # rho = .7
x1 <- x2 <- seq(-2.5, 2.5, length = 51)
z <- matrix(mvtnorm::dmvnorm(expand.grid(x1, x2), mean = c(0,0),
            sigma = Sigma), 51)
par(mfrow = c(1, 2), pty = "s", mai = c(.5, .6, .3, .1))
persp(x1, x2, z, theta = 0, phi = 40, expand = .7,
      col = "lightblue", ltheta = 90, shade = .7,
      ticktype = "detailed", xlab = "x1", ylab = "x2",
      zlab = "Density", main = "Density Function")
contour(x1, x2, z, xlab = "x1", ylab = "x2", mgp = c(2,.7,0),
        main = "Isodensity Curves")
grid()
```

Multivariate Normal Distribution in R

Sampling from a multivariate normal distribution

```
S <- matrix(c(10, 3, 0,
              3, 2, 0,
              0, 0, 1), 3, 3)
x_bar <- 1:3
X <- MASS::mvrnorm(1000, x_bar, S) # 1000 data points from
                                   # 3d-multivariate normal distribution
head(X)
#             [,1]      [,2]     [,3]
# [1,]  0.3785520 0.2349043 3.062647
# [2,]  1.8730729 1.6588387 2.818568
# [3,] -3.8669220 0.9747584 3.432734
# [4,]  1.4116807 2.0623302 2.326369
# [5,] -0.5523353 2.1854139 2.014931
# [6,]  3.0204520 3.3558775 2.996275
```